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The classical nucleation rate in two dimensions

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Abstract. In many systems in condensed matter physics and quantum field theory, first order phase transitions are initiated by the nucleation of bubbles of the stable phase. In homogeneous nucleation theory the nucleation rate Γ can be written in the form of the Arrhenius law: $\Gamma = \mathcal{A}\mathrm{e}^{-\mathcal{H}_c}$. Here $\mathcal{H}_c$ is the energy of the critical bubble, and the prefactor $\mathcal{A}$ can be expressed in terms of the determinant of the operator of fluctuations near the critical bubble state. In general it is not possible to find explicit expressions for $\mathcal{A}$ and $\mathcal{H}_c$. If the difference η between the energies of the stable and metastable vacua is small, the constant $\mathcal{A}$ can be determined within the leading approximation in η, which is an extension of the "thin wall approximation". We have done this calculation for the case of a model with a real-valued order parameter in two dimensions.

1 Introduction

The problem of the decay of the metastable false vacuum at first order phase transitions has attracted considerable interest due to its numerous relations with condensed matter physics [1], quantum fields [2], cosmology [3], and black hole theory [4]. In Langer's theory of homogeneous nucleation [5,6], the false vacuum decay is associated with the spontaneous nucleation of a critical bubble of a stable phase in a metastable surrounding. In the context of quantum field theory, the nucleation theory was developed by Voloshin et al. [7], and Callan and Coleman [8,9]. The quantity of main interest is the nucleation rate I per time and volume.

The nucleation rate

$$I = \frac{\kappa}{2\pi}\Gamma \tag{1}$$

is a product of the static part Γ and the so-called kinetic prefactor κ, which depends on the detailed nonequilibrium dynamics of the model; see [6,1]. Most important is the static nucleation rate Γ, which is equal to twice the imaginary part of the free energy density of the metastable phase. In this article we study the static part Γ.

In the homogeneous nucleation theory it has the form of the Arrhenius law:

$$\Gamma = \mathcal{A}\exp(-\mathcal{H}_c), \tag{2}$$

where $\mathcal{H}_c$ is the energy of the critical bubble. The prefactor $\mathcal{A}$ is determined by fluctuations near the critical bubble state and can be expressed in terms of the functional determinant of the fluctuation operator [5,9].

In the general case, it is not possible to find the explicit critical bubble solution of the field equations analytically. However, the problem becomes asymptotically solvable, if the decaying metastable state is close enough in energy to the stable one, i.e. if the energy density difference η between the metastable and stable vacua is small. The leading approximation in this small parameter is usually called the "thin wall approximation" [10], since at $\eta \to 0$ the critical bubble radius goes to infinity and becomes much larger than the thickness of the bubble wall.

In the thin wall approximation, the critical bubble energy $\mathcal{H}_c$ can be easily obtained from Langer's nucleation theory. It turns out to be much more difficult to find explicitly the prefactor $\mathcal{A}$ in (2). This problem, which is important for applications of nucleation theory, has been extensively studied in different models.

A remarkable result on this subject was obtained by Voloshin [11]. He considered scalar field theory in two dimensions with a potential $U(\phi)$ of the type shown in Fig. 1.

Voloshin claimed that in the limit $\eta \to 0$ the nucleation rate Γ in such a model can be described by the simple universal formula

$$\Gamma = \frac{\eta}{2\pi}\exp\left(-\frac{\pi\sigma^2}{\eta}\right). \tag{3}$$

Here σ is the surface tension of the wall between the stable and metastable vacua in the limit $\eta \to 0$. Thus, according to [11], in this limit the nucleation rate Γ is determined by two well defined macroscopic parameters η and

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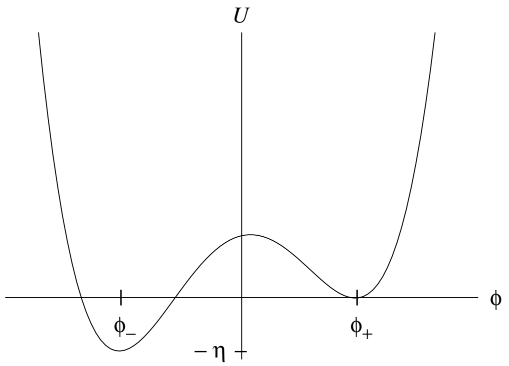

Fig. 1. The potential U with the false (ϕ_+) and true (ϕ_-) vacuum

σ. Another claim of [11] is that there are no corrections to (3) proportional to powers of the dimensionless parameter η/σ^2. Voloshin arrived at these conclusions by an analysis performed in the thin wall approximation. He replaced the original scalar field theory by an effective geometrical one, which describes only fluctuations of the critical bubble shape. This approach implies that all other fluctuations of the original scalar field could be properly accounted for by the correct choice of the macroscopic parameters η and σ.

Recently an analytical method was developed [12], which allows one to study nucleation in the scalar field model beyond the thin wall approximation. In [12] this method was used to calculate the nucleation rate for the first order phase transition in the three-dimensional Ginzburg–Landau model. In the present paper we apply the same approach to the two-dimensional case. We calculate the nucleation rate beyond the thin wall approximation and verify directly Voloshin's claim (3).

Nucleation theory in two-dimensional scalar field theory has also been studied by Kiselev and Selivanov [13], Strumia and Tetradis [14], and other authors. In these articles, however, different renormalization schemes have been used and Γ has not been expressed in terms of macroscopic parameters η and σ. This makes it difficult to compare their results with the ones discussed in this article.

In the articles [15,16] the nucleation rate was calculated in the two-dimensional Ising model in a small magnetic field for arbitrary anisotropies. If Voloshin's result (3) is universal, it should be applicable as well to the Ising model in the critical region. Indeed, the results of [15,16], rewritten in terms of η and σ, are in a very good agreement with (3). The exponent factors are the same, and the prefactors differ only by the number $\pi^2/9 \approx 1.0966$, which is very close to unity. This small discrepancy increased our interest in the subject of the present study.

2 Model and notation

We consider the two-dimensional asymmetric Ginzburg–Landau model defined by the Hamiltonian

$$\mathcal{H}(\phi) = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu \phi(x))^2 + U(\phi(x))\right], \qquad (4)$$

where $\phi(x)$ is the continuous one-component order parameter, and the potential $U(\phi)$ depicted in Fig. 1 is given by

$$U(\phi) = U_\mathrm{s}(\phi) + \frac{\eta_0}{2v}(\phi - v) + U_0. \qquad (5)$$

Here $U_\mathrm{s}(\phi)$ denotes the symmetric part of the potential:

$$U_\mathrm{s}(\phi) = \frac{g}{4!}\left(\phi^2 - v^2\right)^2. \qquad (6)$$

The potential $U(\phi)$ has a metastable minimum (false vacuum) at $\phi = \phi_+$ and a stable one (true vacuum) at $\phi = \phi_-$. The constant term U_0 in (5) is chosen to ensure $U(\phi_+) = 0$.

The partition function is given by the functional integral

$$Z = \int D\phi \exp\left[-\mathcal{H}(\phi)\right]. \qquad (7)$$

The temperature has been absorbed into $\mathcal{H}$.

It is convenient to define the mass m and the inverse coupling parameter β by

$$m^2 = \frac{\partial^2}{\partial \phi^2} U_\mathrm{s}(\phi)\bigg|_{\phi = v} = \frac{gv^2}{3}, \quad \beta = \frac{3m^2}{g}, \qquad (8)$$

and to introduce dimensionless quantities

$$\tilde{x}_\mu = \frac{m}{2}x_\mu, \quad \tilde{\eta} = \frac{g}{2m^4}\eta_0, \quad \varphi(\tilde{x}) = \frac{\phi(x)}{v},$$
$$\varphi_\pm = \frac{\phi_\pm}{v}, \quad \tilde{\mathcal{H}} = \frac{\mathcal{H}}{\beta}. \qquad (9)$$

In dimensionless variables the Hamiltonian and partition function take the form

$$\tilde{\mathcal{H}}(\varphi) = \int \mathrm{d}^2\tilde{x} \left[\frac{1}{2}(\nabla\varphi)^2 + \tilde{U}(\varphi(\tilde{x}))\right], \qquad (10)$$

where

$$\tilde{U}(\varphi) = \frac{1}{2}\left[\left(\varphi^2 - 1\right)^2 - \left(\varphi_+^2 - 1\right)^2\right] + \frac{4}{3}\tilde{\eta}\left(\varphi - \varphi_+\right), \qquad (11)$$

and

$$Z = \int D\varphi(\tilde{x}) \exp\left[-\beta\tilde{\mathcal{H}}(\varphi)\right]. \qquad (12)$$

3 The critical bubble solution

The uniform solutions of the field equation

$$\delta\tilde{\mathcal{H}}/\delta\varphi(\tilde{x}) = 0 \qquad (13)$$

are the stable φ_- and false (metastable) φ_+ vacua given by

$$\varphi_\pm = \pm 1 - \frac{\tilde{\eta}}{3} \mp \frac{\tilde{\eta}^2}{6} - \frac{4\tilde{\eta}^3}{27} + O(\tilde{\eta}^4). \qquad (14)$$

The critical bubble $\varphi_b(\tilde{x})$ is the non-uniform radially symmetric solution of (13) approaching the false vacuum at infinity. That is,

$$-\frac{\mathrm{d}^2\varphi_b}{\mathrm{d}\tilde{r}^2} - \frac{1}{\tilde{r}}\frac{\mathrm{d}\varphi_b}{\mathrm{d}\tilde{r}} + 2\varphi_b(\varphi_b^2 - 1) + \frac{4}{3}\tilde{\eta} = 0,$$
$$\lim_{\tilde{r}\to\infty} \varphi_b(\tilde{r}) = \varphi_+, \qquad (15)$$

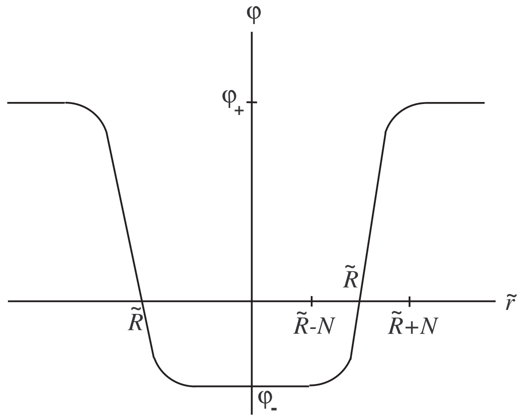

Fig. 2. Profile of the critical bubble

where $\tilde{r} = (\tilde{x}_\mu \tilde{x}_\mu)^{1/2}$. The profile of the critical bubble solution is shown schematically in Fig. 2. If $\tilde{\eta}$ is small, the thin wall centered at $\tilde{r} = \tilde{R}$ divides regions of false and stable vacua outside and inside the bubble, respectively.

Equation (15) cannot be solved explicitly. Following the approach introduced in [12] we shall construct the solution by expansion in powers of $\tilde{\eta}$. Introducing the new independent variable ξ:

$$\xi = \tilde{r} - \tilde{R}, \tag{16}$$

we expand $\tilde{R}$ and $\varphi_b(\xi)$ as

$$\tilde{R} = \frac{a_{-1}}{\tilde{\eta}} + a_0 + a_1\tilde{\eta} + a_2\tilde{\eta}^2 + O(\tilde{\eta}^3), \tag{17}$$

$$\varphi_b(\xi) = \varphi_0(\xi) + \varphi_1(\xi)\tilde{\eta} + \varphi_2(\xi)\tilde{\eta}^2 + O(\tilde{\eta}^3). \tag{18}$$

After substitution of (16–18) into (15) one obtains perturbatively in $\tilde{\eta}$

$$a_{-1} = \frac{1}{2}, \quad a_0 = 0, \quad a_1 = -\frac{2}{9}, \quad a_2 = 0, \tag{19}$$

$$\varphi_0(\xi) = \tanh\xi, \quad \varphi_1(\xi) = -\frac{1}{3},$$

$$\varphi_2(\xi) = -\frac{1}{24\cosh^2\xi} \times \Big\{ 10\xi - 16\xi\cosh(2\xi) - 2\xi\cosh(4\xi) + 2\ln[2\cosh\xi]\,[12\xi + 8\sinh(2\xi) + \sinh(4\xi)] - 24\int_0^\xi \mathrm{d}t\, t\tanh t \Big\}.$$

The bubble energy $\tilde{E} = \tilde{\mathcal{H}}[\varphi_b(x)]$ can be written as

$$\tilde{E} = \pi \int_{-\tilde{R}}^{\infty} \mathrm{d}\xi (\tilde{R} + \xi) \left(\frac{\mathrm{d}\varphi_b(\xi)}{\mathrm{d}\xi} \right)^2. \tag{20}$$

Substitution of (19) into (20) yields

$$\tilde{E} = \frac{2\pi}{3} \left[\frac{1}{\tilde{\eta}} + \tilde{\eta} \left(\frac{19}{18} - \frac{\pi^2}{3} \right) + O(\tilde{\eta}^3) \right]. \tag{21}$$

It is the basic principle of homogeneous nucleation theory that the decay of the metastable vacuum occurs through nucleation of the critical bubble. Callan and Coleman expressed the nucleation rate Γ of the metastable vacuum in terms of functional determinants [8,9]. In our notation their result takes the form

$$\tilde{\Gamma} = \frac{\beta\tilde{E}}{2\pi} \frac{1}{\sqrt{|\lambda_0|}} \exp\left(-\beta\tilde{E} + S \right). \tag{22}$$

Here $\tilde{\Gamma} = 4\Gamma/m^2$ is the dimensionless nucleation rate, and the entropy S associated with the critical bubble is given by

$$\exp S = \left[\frac{\det' M}{\det M^{(0)}} \right]^{-1/2}, \tag{23}$$

where M and $M^{(0)}$ are the fluctuation operators near the bubble $\varphi_b(\tilde{x})$ and the metastable uniform vacuum φ_+, respectively:

$$M = -\partial^2 + 6[\varphi_b(\tilde{r})]^2 - 2, \tag{24}$$

$$M^{(0)} = -\partial^2 + 6\varphi_+^2 - 2. \tag{25}$$

The operator M has two zero modes proportional to $\partial_\mu \varphi_b(\tilde{x})$, $\mu = 1, 2$, and one negative mode with the eigenvalue

$$\lambda_0 = -4\tilde{\eta}^2. \tag{26}$$

The notation det′ implies that the three above mentioned modes are omitted in the corresponding determinant. After substitution of (21) and (26), (22) simplifies to

$$\tilde{\Gamma} = \frac{\beta}{6\tilde{\eta}^2} \exp\left(-\frac{2\pi\beta}{3\tilde{\eta}} + S \right) (1 + O(\tilde{\eta})). \tag{27}$$

In the subsequent sections we shall calculate the small $\tilde{\eta}$ expansion for the critical bubble entropy (23) to the order $O(\tilde{\eta}^0)$.

4 The bubble entropy

The spectrum of the fluctuation operator M can be determined in the form of a perturbative expansion in powers of the parameter $\tilde{\eta}$ as in [12]. This is achieved in the following way.

Introducing the angular momentum quantum number $\mu \in \mathbf{Z}$ in two dimensions, the radial Schrödinger operators corresponding to M and $M^{(0)}$ are

$$H_\mu = -\frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} - \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}} + \frac{\mu^2}{\tilde{r}^2} + 6\,[\varphi_b(\tilde{r})]^2 - 2, \tag{28}$$

$$H_\mu^{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}\tilde{r}^2} - \frac{1}{\tilde{r}}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}} + \frac{\mu^2}{\tilde{r}^2} + 6\,\varphi_+^2 - 2. \tag{29}$$

Shifting the coordinate from $\tilde{r}$ to ξ and making use of the Laurent series (17) for $\tilde{R}$, the eigenvalue problem for H_μ can be treated perturbatively in $\tilde{\eta}$. The lowest order leads to the exactly solvable Pöschl–Teller operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 6\mathrm{sech}^2\xi + 4, \tag{30}$$

which has discrete eigenvalues 0 and 3 and a continuum above 4. In second order one finds a band around 0:

$$\lambda_{0\mu} = 4\tilde{\eta}^2(\mu^2 - 1) + O(\tilde{\eta}^4), \tag{31}$$

a band around 3:

$$\lambda_{3\mu} = 3 + 4\tilde{\eta}^2(\mu^2 + \text{const.}) + O(\tilde{\eta}^4), \tag{32}$$

and a continuum

$$\lambda_{k\mu} = k^2 + 6\varphi_+^2 - 2 + 4\tilde{\eta}^2\mu^2 + O(\tilde{\eta}^4), \tag{33}$$

with $k \in \mathbf{R}$. This spectrum includes the negative mode λ_{00} and the two zero modes $\lambda_{0,\pm 1}$.

The sum over μ and the integration over k produce ultraviolet divergencies in S. We treat these by means of dimensional regularization in $d = 2 - \varepsilon$ dimensions. As there appear volume integrals in intermediate steps of the calculation, the extra dimensions are equipped with a finite extent L and periodic boundary conditions. The parameter L must cancel out in finite results.

The finite part of the regularized entropy can be conveniently evaluated with the help of zeta-function techniques [17]. The operator-zeta function appropriate for our case is defined by

$$\zeta_M(z) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t t^{z-1} \left(\mathrm{Tr}'\mathrm{e}^{-tM} - \mathrm{Tr}\mathrm{e}^{-tM^{(0)}}\right) \tag{34}$$

for $\mathrm{Re} z > 1$ and analytical continuation to other values of z. The integrand contains the heat kernels $\exp(-tM)$ and $\exp(-tM^{(0)})$. For positive t there is an asymptotic expansion, the so-called Seeley expansion, which is of the form

$$\mathrm{Tr}\left(\mathrm{e}^{-tM} - \mathrm{e}^{-tM^{(0)}}\right) = (4\pi t)^{-d/2} \sum_{n=1}^{\infty} t^n \mathcal{O}_n. \tag{35}$$

Following [17] one obtains

$$S = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\zeta_M(0) + \frac{\mathcal{O}_1}{8\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1)\right] + O(\varepsilon). \tag{36}$$

This expression displays the divergence as a simple pole in ε. The derivative of the zeta function is a finite quantity. The first Seeley coefficient is given by

$$\mathcal{O}_1 = -6 \int \mathrm{d}^d\tilde{x}([\varphi_b(\tilde{r})]^2 - \varphi_+^2) = \tilde{L}^{-\varepsilon}\frac{10\pi}{\tilde{\eta}} + O(\tilde{\eta}), \tag{37}$$

where

$$\tilde{L} = \frac{m}{2}L. \tag{38}$$

The zeta function is decomposed into a contribution from the band near zero and the rest,

$$\zeta_M(z) = \zeta_0(z) + \zeta_1(z), \tag{39}$$

where

$$\zeta_0(z) = \frac{1}{\Gamma(z)} \int_0^\infty \mathrm{d}t t^{z-1} \sum_{\mu \neq 0, \pm 1} \mathrm{e}^{-t\lambda_{0\mu}}. \tag{40}$$

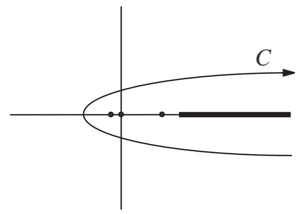

Fig. 3. Integration path C in the complex λ-plane

Correspondingly, the entropy is decomposed as

$$S = S_0 + S_1 + \tilde{L}^{-\varepsilon}\frac{5}{4\tilde{\eta}}\left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1)\right] + O(\tilde{\eta}) + O(\varepsilon). \tag{41}$$

Consider the part

$$S_0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\zeta_0(0). \tag{42}$$

For a general spectrum of the type

$$\lambda_\mu = a(\mu + b)(\mu + c) \tag{43}$$

the logarithm of the zeta-function regularized determinant is given by

$$-\frac{\mathrm{d}}{\mathrm{d}z}\zeta_0(0) = -2\ln\Gamma(b+1) - 2\ln\Gamma(c+1) - (b+c)\ln a + \ln(bc) + 2\ln(2\pi), \tag{44}$$

which can be derived with the help of Riemann's and Hurwitz's zeta functions. Setting $a = 4\tilde{\eta}^2, b = 0, c = 2$ one finds

$$S_0 = \ln\left(\frac{8\tilde{\eta}^3}{\pi}\right) + O(\tilde{\eta}^2). \tag{45}$$

This piece will therefore contribute the factor

$$\mathrm{e}^{S_0} = \frac{8\tilde{\eta}^3}{\pi}\{1 + O(\tilde{\eta}^2)\} \tag{46}$$

to the prefactor $\mathcal{A}$ of Γ.

The remaining part S_1 of the entropy is calculated with the help of methods from quantum mechanical scattering theory. The heat kernels can be represented as

$$K_t(M) \equiv \mathrm{Tr}\left(\mathrm{e}^{-tM} - \mathrm{e}^{-tM^{(0)}}\right) = -\int_C \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\,\mathrm{e}^{-\lambda t}\mathrm{Tr}\left[(\lambda - M)^{-1} - (\lambda - M^{(0)})^{-1}\right], \tag{47}$$

where the integration path C in the complex plane is shown in Fig. 3.

Decomposed into the angular momentum sums this reads

$$K_t(M) = -\sum_\mu \int_C \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\mathrm{e}^{-\lambda t}A(\lambda,\mu), \tag{48}$$

with

$$A(\lambda,\mu) = \mathrm{Tr}\left[(\lambda - H_\mu)^{-1} - (\lambda - H_\mu^{(0)})^{-1}\right]. \tag{49}$$

We obtained an exact representation for $A(\lambda, \mu)$. To describe it some notations are necessary.

Let $f_i(\tilde{r}, \lambda),\ g_i(\tilde{r}, \lambda),\ i = 1, 2$ be the solutions of the linear ordinary differential equation

$$H_\mu\ \psi(\tilde{r}) = \lambda\ \psi(\tilde{r}) \tag{50}$$

determined by their asymptotics:

$$\begin{aligned} f_1(\tilde{r}, \lambda) &\to K_\mu(q\tilde{r}), \\ &\text{and} \quad f_2(\tilde{r}, \lambda) \to I_\mu(q\tilde{r}), \ \text{at } \tilde{r} \to \infty, \quad (51) \\ g_1(\tilde{r}, \lambda) &\to K_\mu(q_-\tilde{r}), \\ &\text{and} \quad g_2(\tilde{r}, \lambda) \to I_\mu(q_-\tilde{r}), \ \text{at } \tilde{r} \to 0. \quad (52) \end{aligned}$$

Here $K_\mu(z)$ and $I_\mu(z)$ are modified Bessel functions, and the parameters q and q_- are defined as

$$q = \left(6\varphi_+^2 - \lambda - 2\right)^{1/2}, \quad q_- = \left(6\varphi_-^2 - \lambda - 2\right)^{1/2}. \tag{53}$$

Since the second order equation (50) has two linearly independent solutions, there is a linear dependence between the functions $g_i(\tilde{r}, \lambda)$ and $f_i(\tilde{r}, \lambda)$:

$$g_i(\tilde{r}, \lambda) = \sum_{j=1,2} \alpha_{ij}(\lambda) f_j(\tilde{r}, \lambda). \tag{54}$$

The function $A(\lambda, \mu)$ can be expressed explicitly in terms of the coefficient $\alpha_{22}(\lambda)$:

$$A(\lambda, \mu) = \frac{3\mu(\varphi_+^2 - \varphi_-^2)}{(\lambda + 2 - 6\varphi_-^2)(\lambda + 2 - 6\varphi_+^2)} + \frac{\mathrm{d} \ln \alpha_{22}(\lambda)}{\mathrm{d}\lambda}. \tag{55}$$

This representation for the trace of resolvent operators is exact. However, (50) cannot be solved in closed form for arbitrary $\tilde{\eta}$. So we have to consider the small-$\tilde{\eta}$ expansion for $\alpha_{22}(\lambda)$. We have obtained two terms of this expansion by use of a perturbation theoretical analysis of the scattering problem (50–54). Omitting the details, the logarithmic derivative of the matrix element $\alpha_{22}(\lambda)$ up to quadratic terms in $\tilde{\eta}$ takes the form

$$\begin{aligned} \frac{\mathrm{d} \ln \alpha_{22}(\lambda)}{\mathrm{d}\lambda} &= \frac{1}{(4 + p^2 - \lambda)^{1/2}} \qquad (56) \\ &\times \left[\frac{2p^2}{(\lambda - 4)^2} - \frac{1}{\lambda - 4} + \frac{1}{\lambda - 3 - p^2} + \frac{2}{\lambda - p^2} \right] + O(\tilde{\eta}^2). \end{aligned}$$

Here p is the angular momentum parameter defined as $p = 2\tilde{\eta}\mu \approx \mu/\tilde{R}$.

Substitution of (56) and (14) into (55) yields

$$\begin{aligned} A(\lambda, \mu) &= \frac{1}{(4 + p^2 - \lambda)^{1/2}} \\ &\times \left[\frac{2p^2}{(\lambda - 4)^2} - \frac{1}{\lambda - 4} + \frac{1}{\lambda - 3 - p^2} + \frac{2}{\lambda - p^2} \right] \\ &- \frac{2\,|p|}{(\lambda - 4)^2} + O(\tilde{\eta}^2). \qquad (57) \end{aligned}$$

This function has simple poles at $\lambda = p^2, \lambda = 3 + p^2$ and a square root branching at $\lambda = 4 + p^2$. It is analytic at $\lambda = 4$.

With the help of this expression we could evaluate the integral representation for $K_t(M)$ and the related zeta function. The details of this lengthy analysis will not be presented here. The μ-summations have been done by means of Poisson's summation formula. Separating the contribution of the band near zero, which has been treated above, the final result for the remaining entropy is

$$S_1 = \frac{1}{2\tilde{\eta}} \left(6 + \frac{2\pi}{\sqrt{3}} - 5 \ln 2 \right). \tag{58}$$

This is the central result of this section.

5 Decay rate

In $d = 2 - \varepsilon$ dimensions the bubble energy is associated with an additional factor $L^{-\varepsilon}$ and the regularized dimensionless decay rate $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma} = \frac{\beta L^{-\varepsilon}}{6\tilde{\eta}^2} \exp\left(-\frac{2\pi\beta L^{-\varepsilon}}{3\tilde{\eta}} + S \right) (1 + O(\tilde{\eta})). \tag{59}$$

In terms of the dimensionless quartic coupling

$$u = \frac{g}{m^{4-d}} \tag{60}$$

the parameter β is equal to

$$\beta = \frac{3}{u} m^{-\varepsilon}. \tag{61}$$

Using the results (41,46), we can write the nucleation rate as

$$\begin{aligned} \tilde{\Gamma} &= (2\tilde{L}^{-\varepsilon}) \frac{4\tilde{\eta}}{\pi u} \\ &\times \exp\Bigg(- \frac{2\pi}{u\tilde{\eta}} (2\tilde{L})^{-\varepsilon} + \tilde{L}^{-\varepsilon} \frac{5}{4\tilde{\eta}} \left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1) \right] \\ &+ S_1 + O(\tilde{\eta}) + O(\varepsilon) \Bigg). \qquad (62) \end{aligned}$$

The entropy contains an UV-divergent term, represented by a pole in ε. After renormalization of the parameters of the model according to the usual prescriptions, the divergencies as well as the spurious L-dependence should disappear in the limit $\varepsilon \to 0$.

For the renormalization of the model parameters we use the same scheme as in [12,17]. A straightforward calculation yields the relation between the bare and renormalized dimensionless couplings and masses on the one-loop level:

$$u = u_R\left\{1 - \frac{u_R}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1) + \frac{3}{4}\right] + O(\varepsilon) + O(u_R^2)\right\}. \quad (63)$$

$$m^2 = m_R^2\left\{1 + \frac{u_R}{4\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1) + \frac{7}{4}\right] + O(\varepsilon) + O(u_R^2)\right\}. \quad (64)$$

The asymmetry parameter η_0 is renormalized as follows. On tree level the difference between the minima of the potential

$$\Delta U = U(\phi_+) - U(\phi_-) = \eta_0 + O(\eta_0^3) \quad (65)$$

is equal to η_0 for small asymmetries. Let U_{eff} be the full effective potential given by

$$\Gamma[\phi] = -\int \mathrm{d}^2x\ U_{\mathrm{eff}}(\phi) \quad \text{for } \phi = \text{const.}, \quad (66)$$

where $\Gamma[\phi]$ is the generating functional of one-particle irreducible vertex functions. Then we define the renormalized asymmetry parameter η through

$$\eta = \Delta U_{\mathrm{eff}} = U_{\mathrm{eff}}(\langle\phi\rangle_+) - U_{\mathrm{eff}}(\langle\phi\rangle_-). \quad (67)$$

It is related to η_0 by

$$\eta = \frac{\eta_0}{v}\langle\phi\rangle + O(\eta_0^2), \quad (68)$$

where $\langle\phi\rangle$ is the expectation value of the field at $\eta_0 = 0$. From a one-loop calculation we get

$$\eta_0 = \eta\left\{1 + \frac{u_R}{8\pi}\left[\frac{2}{\varepsilon} + \ln 4\pi + \Gamma'(1)\right] + O(\varepsilon) + O(u_R^2)\right\}. \quad (69)$$

Expressing the unrenormalized parameters in terms of their renormalized counterparts, the divergencies cancel indeed and in the limit $\varepsilon = 0$ we obtain

$$\Gamma = \frac{\eta}{2\pi}\mathrm{e}^{-F} \quad (70)$$

with

$$F = -S_1 + 4\pi\frac{m_R^2}{u_R^2\eta} \times \left\{1 + \frac{u_R}{4\pi}\left(\frac{5}{4} - 5\ln 2\right) + O(u_R^2)\right\} + O(\eta) \quad (71)$$

$$= 4\pi\frac{m_R^2}{u_R^2\eta} \times \left\{1 - \frac{u_R}{4\pi}\left(\frac{19}{4} - \frac{\pi}{\sqrt{3}}\right) + O(u_R^2)\right\} + O(\eta). \quad (72)$$

To make contact with Voloshin's proposal we have to express this result in terms of the interface tension σ. To this end we calculated σ along the lines of [17, 18], but now for $d = 2$. Leaving out the details here, we obtained

$$\sigma = \frac{2m_R}{u_R}\left\{1 - \frac{u_R}{8\pi}\left(\frac{19}{4} - \frac{\pi}{\sqrt{3}}\right) + O(u_R^2)\right\}. \quad (73)$$

This implies

$$F = \frac{\pi\sigma^2}{\eta}\left\{1 + O(u_R^2)\right\} + O(\eta). \quad (74)$$

Our final result is therefore

$$\Gamma = \frac{\eta}{2\pi}\exp\left(-\frac{\pi\sigma^2}{\eta}\left\{1 + O(u_R^2)\right\} + O(\eta)\right). \quad (75)$$

This is in perfect agreement with Voloshin's result.

Studies of the nucleation rate in the two-dimensional Ising model have been made with the Monte Carlo method; see e.g. [19, 20, 1]. We would like to add a remark on this case. Let

$$Z = \sum_{\{S\}} \exp\left\{K\sum_{\langle ij\rangle} S_iS_j + H\sum_i S_i\right\} \quad (76)$$

be the partition function for the two-dimensional Ising model on a square lattice with lattice spacing 1. The critical coupling is

$$K_c = \frac{1}{2}\ln(\sqrt{2} + 1). \quad (77)$$

The quantities appearing in (75) can be related exactly to K and H in the critical region. The interface tension is given by [21]

$$\sigma = 2K + \ln\tanh K, \quad (78)$$

and the asymmetry parameter is

$$\eta = 2MH, \quad (79)$$

where

$$M = \left(1 - [\sinh 2K]^{-4}\right)^{1/8} \quad (80)$$

is the zero field magnetization [22].

The kinetic prefactor κ, mentioned in the introduction, cannot be calculated with static methods, because it depends on the non-equilibrium dynamics. For dynamics that can be described by a Fokker–Planck equation, it is, however, expected to be proportional to the negative eigenvalue $|\lambda_0|$ and contributes two additional powers of the magnetic field [6, 1].

6 Summary

Our semiclassical calculation of the nucleation rate Γ in the two-dimensional Landau–Ginzburg ϕ^4-model confirms Voloshin's result (3), which was derived in the thin wall approximation. In particular, we confirm the prefactor value $\mathcal{A} = \eta/(2\pi)$ first obtained by Kiselev and Selivanov [13], and Voloshin [11].

This value differs from that obtained for the two-dimensional critical Ising model [15,16] by the numerical factor $\pi^2/9 \approx 1.0966$. We suppose that this small discrepancy is the result of approximations used in [15,16], and the prefactor value $\mathcal{A} = \eta/(2\pi)$ is universal. This statement was supported also in the numerical calculations of $\mathcal{A}$ in the critical Ising model by Fonseca and Zamolodchikov [23].

The nucleation rate (75) obtained for the model in the continuum indicates no oscillations in $1/\eta$, in contrast to the analogous expression (35) of [16] for the Ising model on a lattice. This supports the assumption on the discrete-lattice nature of the oscillations in $\Gamma(H)$ observed in the Ising model [16].

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